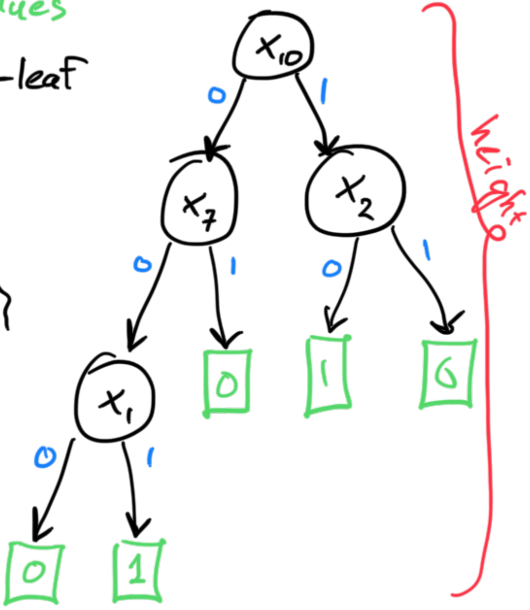
Decision trees

Def: DT is a binary tree

- internal nodes labelled with input vars
- leaves labelled with output values

Each $x \in \{0,1\}^n$ defines root-to-leaf path $\rightarrow$ output value of leaf

↳ DT computes $f: \{0,1\}^n \rightarrow \{0,1\}$

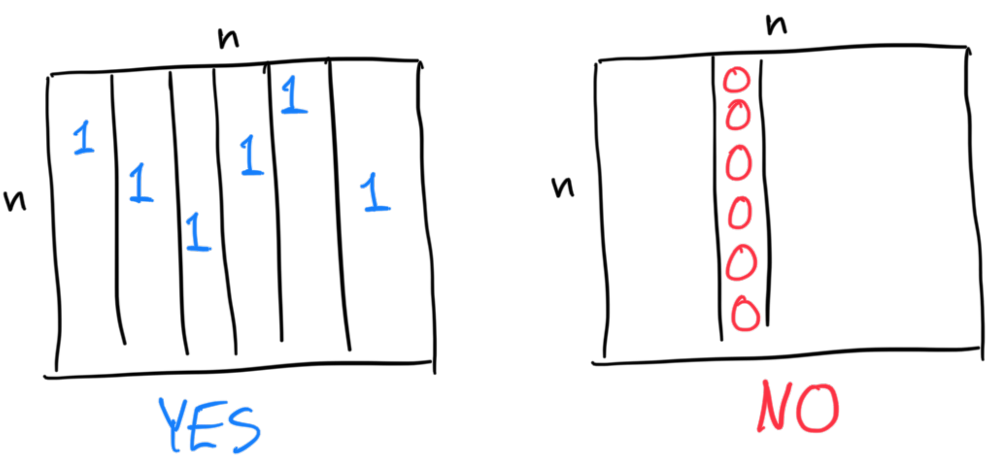


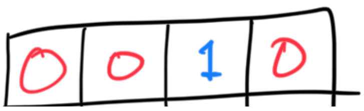
$D(f)$ = least height of DT computing f

Ex: $D(OR_n) = n$ (Recall $P^A \neq NP^A$)

Note: Can prove lower bounds on $D(f)$ using Adversary argument (Exercise: characterises $D(\cdot)$)

Ex: $D(AND_n \circ OR_n) \overset{?}{=} n^2$



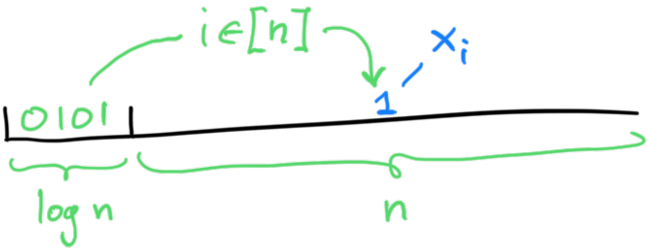


0	0	0	0
0	1	0	0
1	0	0	$\boxed{1/0}$

Ex: Index: $\{0,1\}^{\log n + n} \to \{0,1\}$

$(i, x) \mapsto x_i$

$D(\text{Ind}) = \log n + 1$



Def: Fix $f: \{0,1\}^n \to \{0,1\}$.

Partial assignment $\rho \in \{0,1,*\}^n$ is a **certificate** for $x \in \{0,1\}^n$ if

- ρ consistent with x $\quad (\rho_i = x_i \ \forall i,\ \rho_i \neq *)$
- if ρ consistent with x' $\Rightarrow f(x') = f(x)$

Ex: $f = OR_n$ $\qquad x = 0011 \qquad \rho = **1*\ ,\ |\rho| = 1$

$\qquad\qquad x = 0000 \qquad \rho = 0000\ ,\ |\rho| = n$

Def:
- $C(f, x) =$ least size of cert for x.
- $C_{1/0}(f) = \max\limits_{x : f(x) = 1/0} C(f, x)$ $\quad$ ("NP" and "coNP")
- $C(f) = \max\limits_{x} C(f, x) = \max\{C_1(f), C_0(f)\}$

Ex $\quad C_1(OR_n) = 1, \quad C_0(OR_n) = n$

$C(AND_n \circ OR_n) = n \ll n^2 = D(\text{AndOR})$

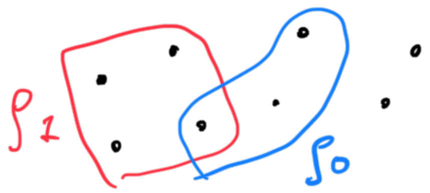
Claim: $C(f) \le D(f)$

Note: $C_1(f)$ = least k s.t. f can be written as k-DNF

$f = \bigvee_i T_i$ width-k terms = size-k cert

Thm: $D(f) \leq C_1(f)\, C_0(f)$ $\left(\leq C(f)^2\right)$

Proof Key obs: every 1-cert ρ_1 intersects every 0-cert ρ_0 (among read bits)



First round: Pick some ρ_1 and query its non-$*$ bits, getting some answer σ

$$\rho_1 = * * 0 1 *$$
$$\sigma = * * 1 1 *$$

Define f_σ as f but with σ values substituted in

$\Rightarrow$ $C_1(f_\sigma) \leq C_1(f)$

$C_0(f_\sigma) \leq C_0(f) - 1$ all 0-certs have shrunk

$\Rightarrow$ Repeat recursively $\square$

Def: Sensitivity:

x but x_i flipped

$$s(f,x) = \left|\{ i : f(x) \neq f(x^{(i)}) \}\right|$$

$$s(f) = \max_x s(f,x)$$

Ex. $s(OR_n) = n$ $\quad (x = 0000)$

$s(AND_n \circ OR_n) = n$ $\quad \mapsto 1$

Claim: $s(f) \leq C(f)$

Say $s(f) = s(f,x)$ $\qquad x = \overbrace{\underbrace{01101}_{s}}^{\rho}001$

Sensitivity conjecture:

Does $D(f) \leq s(f)^k$ for some $k \in \mathbb{N}$?

Open for 30 years....

Then [Huang 19]: Yes! (Annals of Math)

Weaker thm: $C(f) \leq s(f) \cdot bs(f)$ — block sens.

$bs(f,x) =$ max number k of disjoint $B_1, \dots, B_k \subseteq [n]$ s.t. $f(x) \neq f(x^{B_i})$ — flip whole block B_i

$x = 01\overbrace{101}^{B}$

$x^B = 01010$

Proof of $C(f) \leq s(f) \cdot bs(f)$

Let $x \in \{0,1\}^n$, say $f(x) = 1$

max collection of blocks, each block minimal

$$x = 0 \; \overbrace{1 \; 1}^{B_1} \; 0 \; \overbrace{1 \; 1 \; 0}^{B_1} \; 0 \mapsto f(x) = 1$$

$$\rho = * \; 1 \; 1 \; * \; 1 \; 1 \; 0 \; *$$

$$\left[x' = 0 \; 1 \; 1 \; 1 \; 1 \; 1 \; 0 \; 1 \mapsto f(x') = 0 \right]$$

B_3

Show: $|B_i| \leq s(f)$